The visible and the invisible in mathematics education research: tales of arguments, signs, and disciplinarities

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The problematique

- Mathematics education research concerns a series of decisions within the complexity of the educational phenomena.
    - Why?
    - Who?
    - Whom?
    - What?
    - How?
    - [...]

The perspective

- The theoretical frameworks and the scientific research methodologies intertwined with the technological means are the lenses that allow us to view the phenomenon we research, and, at the same time, constitute this phenomenon.
    - [constitute the phenomenon we research]

The perspective

- Depending on the theoretical, methodological, and technological tools we choose, we choose which phenomenon we investigate, which aspects of the educational complexity are (directly or indirectly) visible for our research, our scientifically acceptable investigations and inferencing.
- In this talk, I shall consider argumentation, mathematical notation, and digital storytelling as means for highlighting aspects of the role of the theoretical-methodological-technological lenses as constituting factors in mathematics education research.

Acknowledging complexity …

- Complexity
  - Reality, Realities, Phenomenon, Phenomena …
- Perspective
  - Aims/Intentionality
  - Theory & Research question(s)
  - Methods
  - Sample – roles
  - Data
  - Results
  - Conclusions
- But … do we really need a perspective?
  - Position: *The impossibility of viewing (thus researching) from nowhere!*
- Commitment & Consistency

Between the visible and the invisible

- *Reflections about the interactions of the perceived past and future with the purpose to engineer the (arbitrarily defined as) present.*
- Invisible ⇆ Visible
    - within the complexity of the phenomena
- The scientific approach provides the tools that highlight the visible and expose (projections of) the invisible
    - visible-invisible technological-temporal dependent...

Between the visible and the invisible

- *In the interaction and shift between the visible and the invisible, theory, techniques, and technology constitute decisive factors in our grasping aspects of the teaching-learning phenomenon,*
    - *by defining the under-investigation teaching-learning situation*
    - *by identifying through the possibilities of the descriptions and the inferences the phenomenon (incl. actuality, experiences, normativity, intentionalities).*

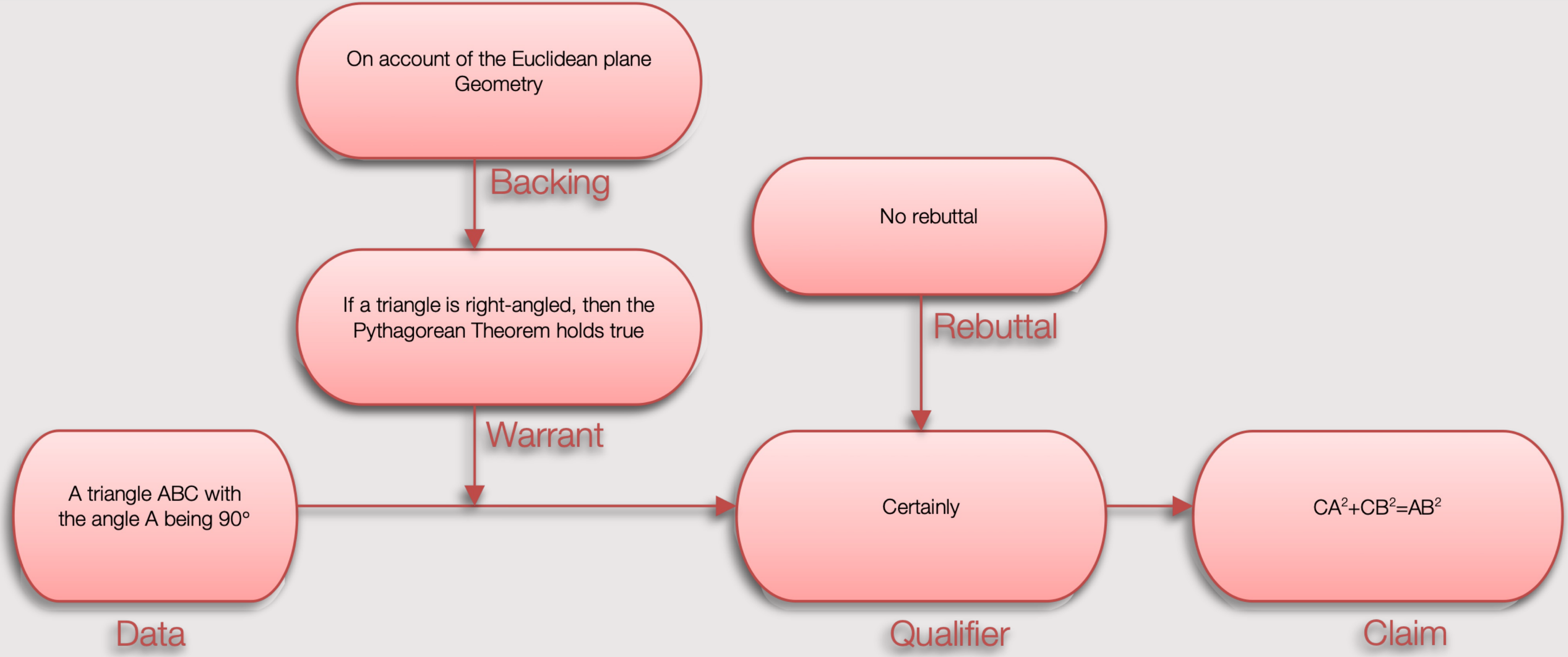
The perspective

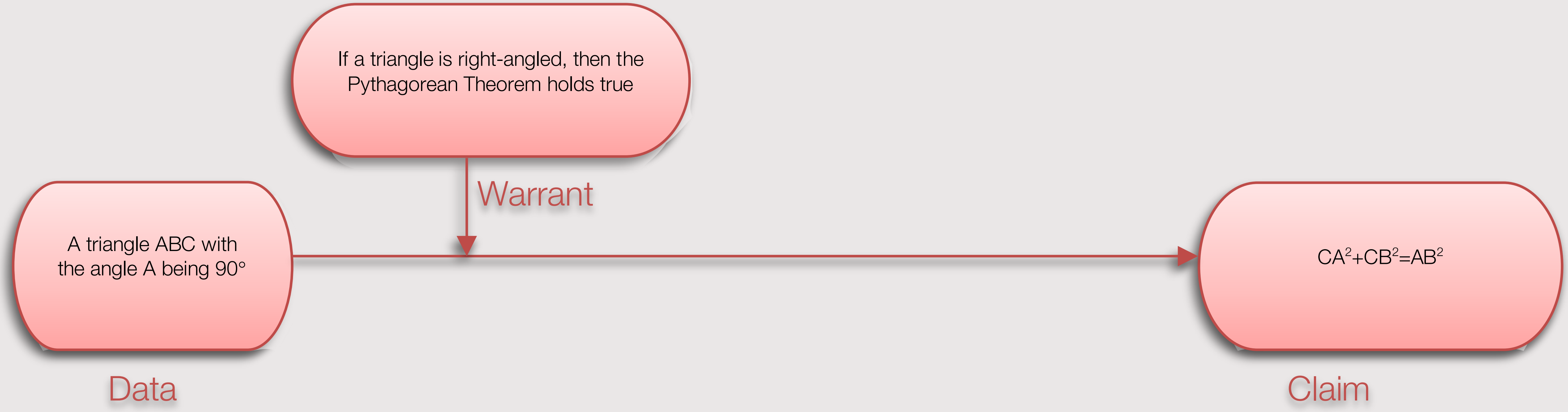
- The theoretical frameworks and the scientific research methodologies intertwined with the technological means are the lenses that allow us to view the phenomenon we research, and, at the same time, constitute this phenomenon.
  - [constitute the phenomenon we research]

Argumentation: tales with Toulmin's scheme

- In *The Uses of Argument*, Toulmin (1958) developed a scheme to analyse the logical micro-structure of an argument.
- According to the scheme, each argument includes Data, Warrant, Backing, Qualifier, Rebuttal, Claim.
  - A Claim is drawn upon some facts (the Data), based on a rule (a hypothetical statement; the Warrant) that associates such facts to this claim.
  - This relationship is valid to a degree of certainty (the Qualifier), unless there is a case of refuting this relationship (the Rebuttal).
  - The applicability of employing a warrant in the specific argument is supported by a categorical statement (the Backing) that identifies the broader system within this warrant may be utilised.


A
90°
C
B
Given the triangle ABC with C = 90°,
then CA²+CB²=AB²


On account of the Euclidean plane Geometry
Backing
If a triangle is right-angled, then the Pythagorean Theorem holds true
Warrant
No rebuttal
Rebuttal
A triangle ABC with the angle A being 90°
Certainly
CA²+CB²=AB²
Data
Qualifier
Claim


If a triangle is right-angled, then the Pythagorean Theorem holds true
Warrant
A triangle ABC with the angle A being 90°
$CA^2+CB^2=AB^2$
Data
Claim

Toulmin's scheme in mathematics education

- Though the scheme was not designed specifically for mathematical arguments, it has been widely utilised in mathematics education research.
  - restricted versions (e.g. Krummheuer, 1995),
  - full version (e.g. Inglis et al., 2007),
  - expanded versions (e.g. Aberdein, 2005; Krummheuer, 1995; Pedemonte & Balacheff, 2016),
  - collective argumentation (e.g. Knipping & Reid, 2013),
  - non-verbal argumentation (e.g. Moutsios-Rentzos, 2022),
  - teachers' supporting the students' mathematical argumentation (Conner et al., 2014),
  - development of teaching tools (e.g. Hein & Prediger, 2017; Moutsios-Rentzos & Micha, 2018)
  - [see also, Cramer & Kempen (2022)]
  - […]

For example … collective argumentation …

- Krummheuer (1995, 2007)

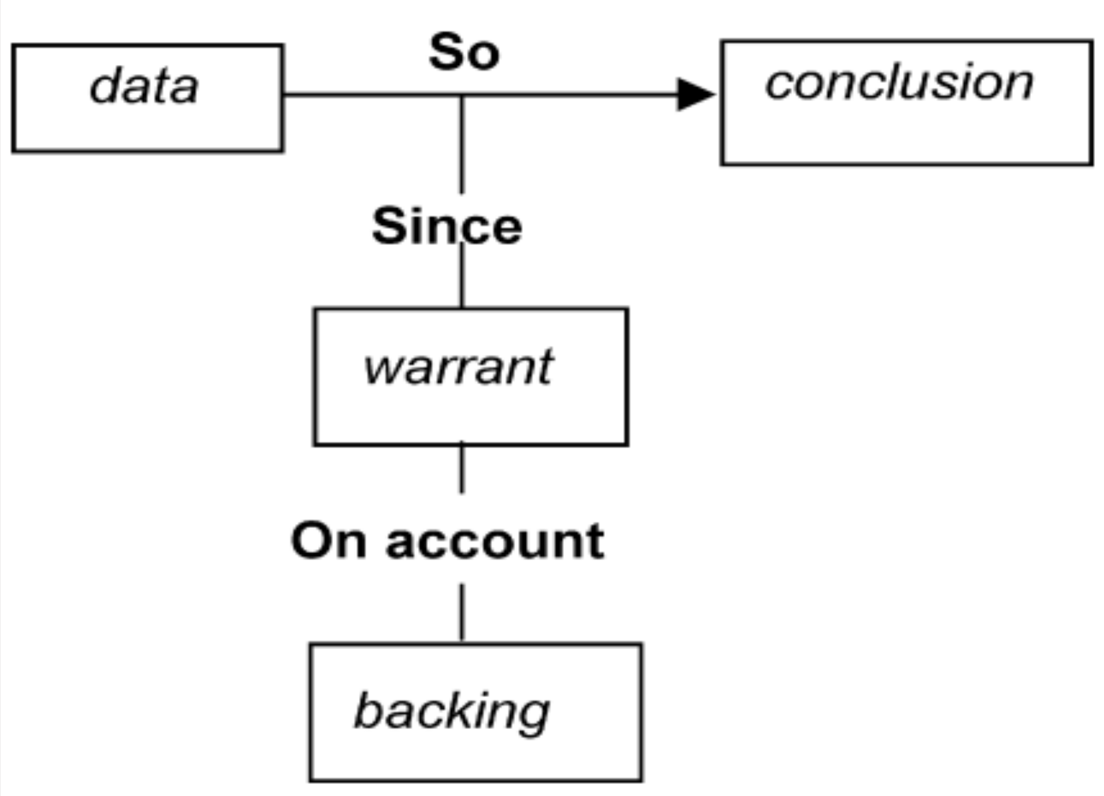


Fig. 1. Toulmin's diagram of argumentation.

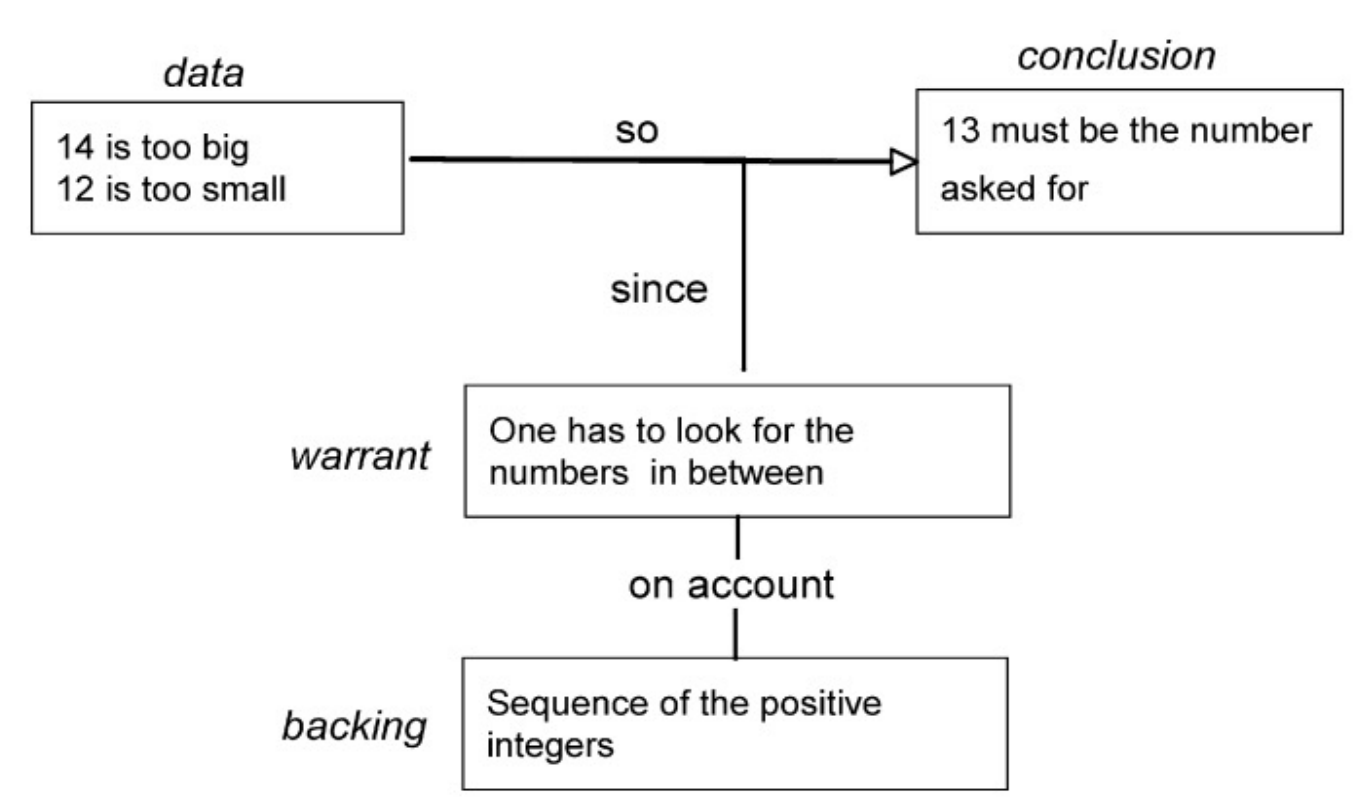
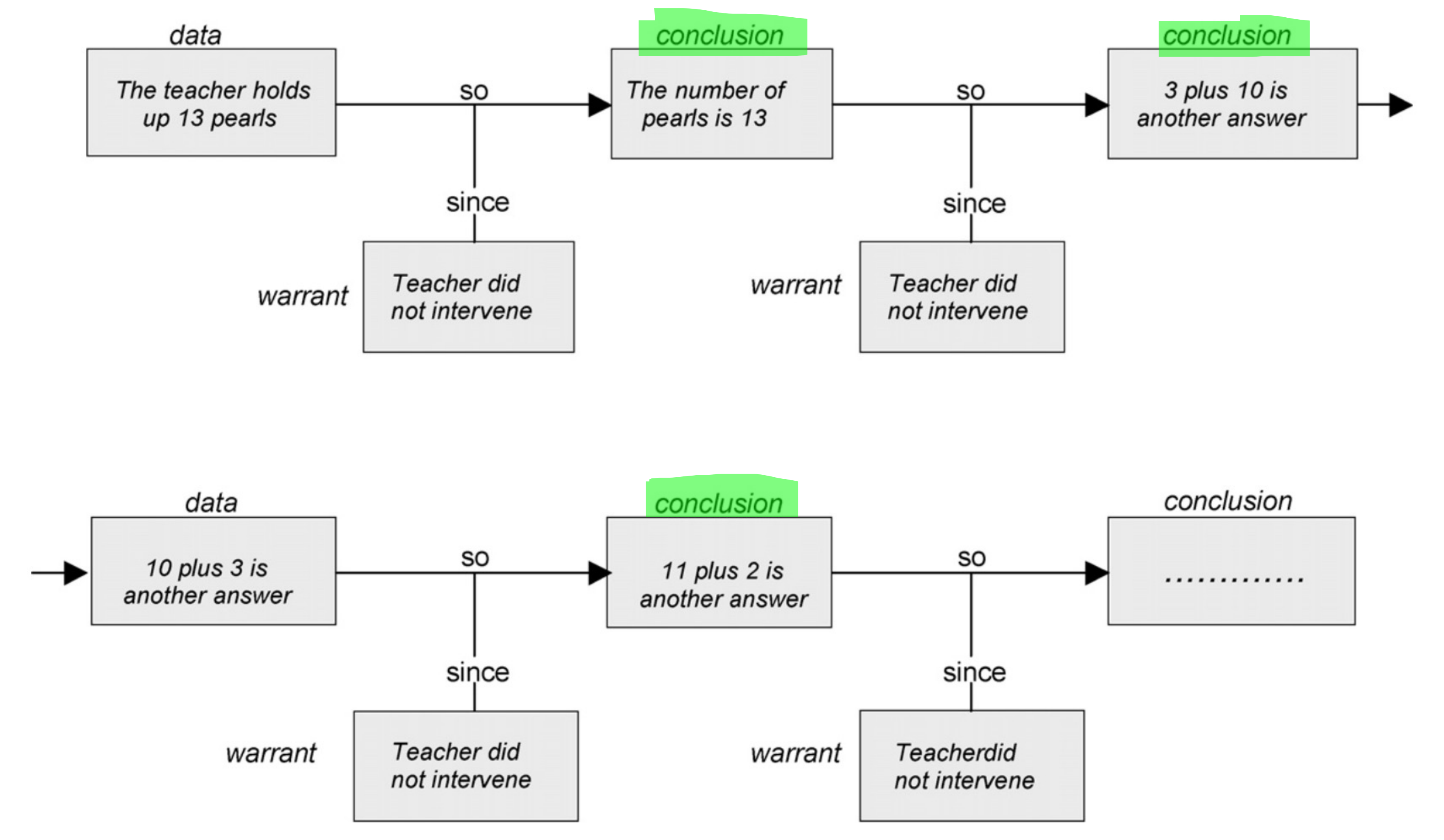


Fig. 8. Complete final argumentation.

A chain of argumentations ...

- (Krummheuer, 2007)



Local and global argumentation structures (Knipping & Reid, 2013)

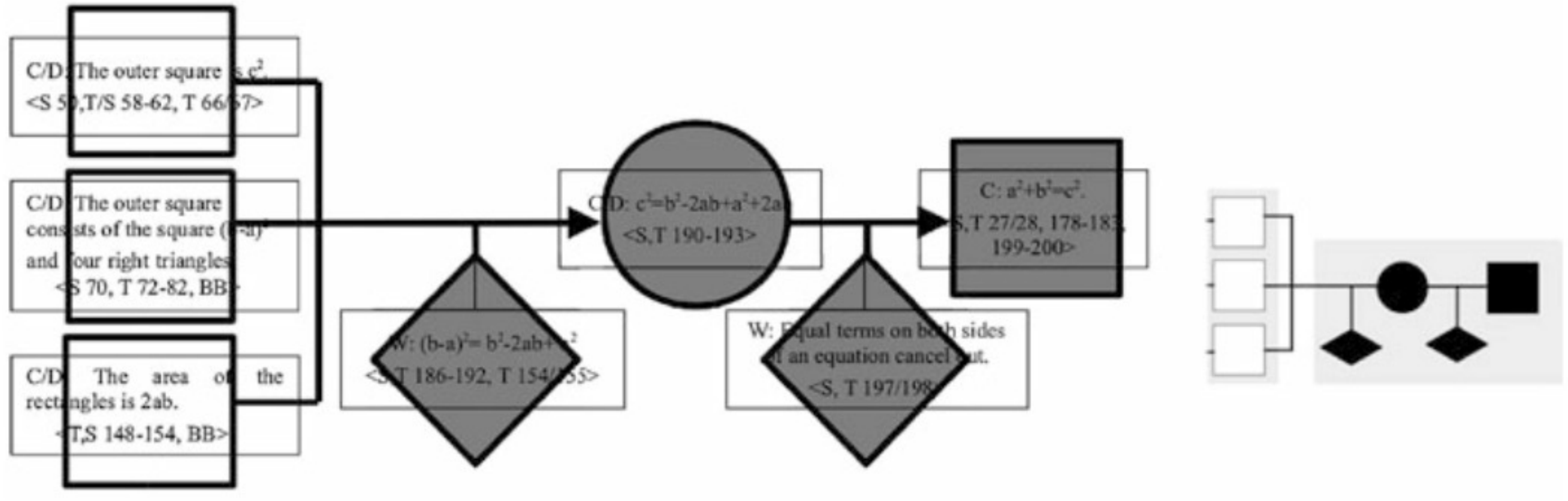
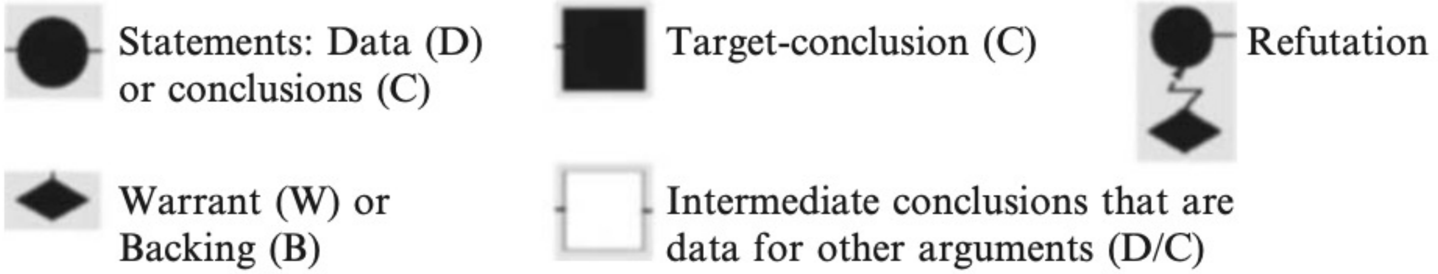


Fig. 8.8 The method of reconstructing a global argumentation



Global argumentation structures (Knipping & Reid, 2013)

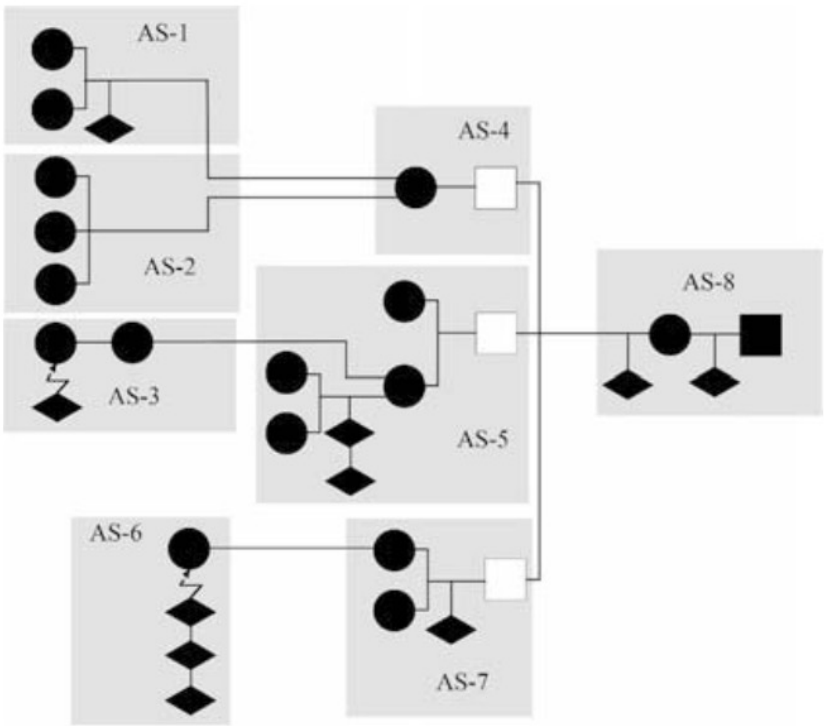


Fig. 8.10a *Source*-structure

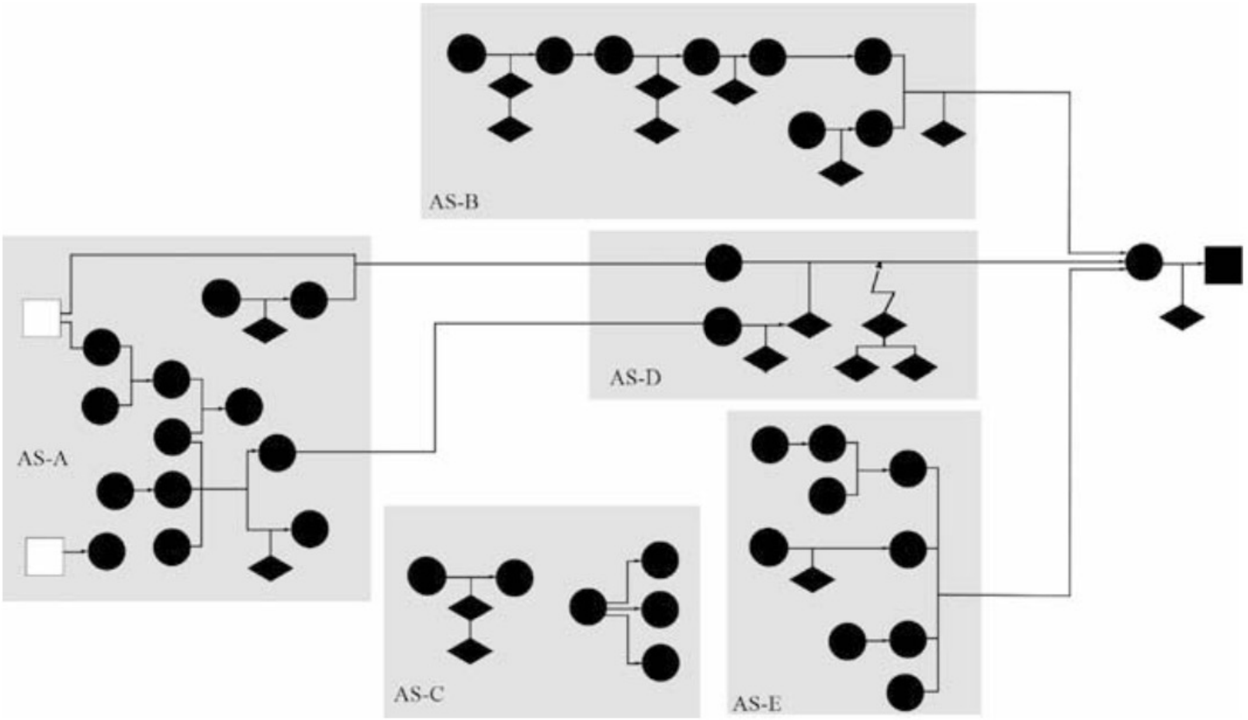


Fig. 8.10b *Spiral*-structure

… extending and/or complementing Toulmin …

- Krummmheuer (2007): The analysis of participation.
  - …mathematical autonomy … responsibility … originality
- Goffman
  - the content-related (semantic) contribution (function of content)
  - the syntactical form with its specific choice of words and its specific formulation (function of formulation)

The responsibilities of the speaking person

	Responsibility for the *content* of an utterance	Responsibility for the *formulation* of an utterance
Author	+	+
Relayer	–	–
Ghostee	+	–
Spokesman	–	+

speaker: function	statement	*reference to a previous speaker*	idea (argumentative function of the statement)
teacher: author	why could it only be thirteen in the end		unambiguousness of the solution number 13. (conclusion)
David: author	because fourteen was too big –		14 was too big. (data)
David: relayer	because fourteen was too big \	*David*	
teacher: spokesman	*points at the 14* yes \	*David*	14 was too big. (data)
teacher: author	and		link to still to be created second data (warrant)
teacher: author	*points at the 12*		12 was too low. (data)
David: spokesman	the . twelve was too small	*teacher*	12 was too small. (data)
Petra: relayer	because cause fourteen was too big / an and twelve was too too small \	*David*	upper and lower limit (data + warrant)
teacher: author	And in between there is only one /		Sequence of positive integers (backing)
Petra: relayer	thirteen	*teacher*	Sequence of positive integers (backing)

… in multimodal argumentation and proving …

- Proof as multimodal text (ProMoTe tool)
- Explicit elements
  - verbal and non-verbal aspects sensory-present
    - natural language, symbols, figures, embodied-sensory (e.g., gestures, posture, voice pitch) etc.
- Implicit elements
  - only mentally present, not consciously present, hypothesised by the analyst.
- Cognitive/affective aspects

(Moutsios-Rentzos, 2022)

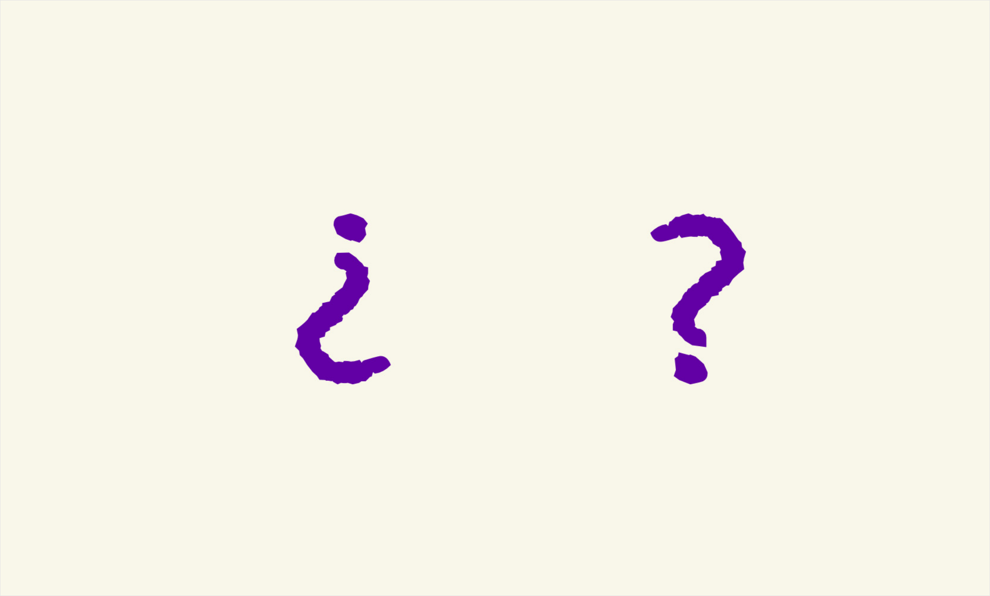
ProMoTe Elements	Data	Warrant	Claim	Qualifier	Rebuttal	Backing
Explicit						
Verbal						
natural language						
symbols (numbers, symbols etc)						
Non-verbal						
written (figures etc)						
embodied -sensory (gestures etc)						
Implicit						
Verbal						
mental (non-explicitly communicated verbal warrants etc)						
Non-verbal						
mental (non-explicitly communicated non-verbal warrants etc)						
embodied -bio-metric (heart-rate etc)						

- A teacher utilizes the example of f(x)=3x to argue that
  - “when α is positive [referring to the general form f(x)=αx], the line goes upwards”.
  - Upwards gestures over the straight line, followed by a movement of the hands from left to right on the horizontal axis
  - The graph of the function f(x)=3x is depicted on the blackboard as a line going upwards, while α=3 is also noted.
- The teacher verbally utters the general rule, whilst verbally and non-verbally notes and shows the specific example on the blackboard.

- Arguments (complementary or antagonistic) developed in parallels:
  - a deductive argument discussing the graph of the specific function as a special case of the general rule he had presented a few moments earlier in the class,
  - an inductive argument about a general rule as generated by the specific example.
- The students are implicitly required to differentiate between these two arguments, and which warrants link which data with which claim.

- “The function y=3x is of the type [raises his voice a little, makes intense hand-movements, his eyes bulging out] … y= αx”
  - Explicitly and implicitly the teacher’s *verbal argumentation* seems to promote deductive argumentation.
  - His implicit and explicit *non-verbal argumentation* appears to legitimatise his authority as means for the validation of mathematical argument, as the embodied-sensory authority-derived warrants together with the deductive implicit warrants are linked with an absolute qualifier, which may communicate *inappropriate* warrant-qualifier pairings as mathematically acceptable (Inglis et al., 2007).

Argumentation: tales with Toulmin's scheme



… beyond Toulmin …

- Toulmin's scheme is *only one* of the diverse perspectives of investigating argumentation
  - full/restricted, extended, or complemented etc.
- e.g., Habermas, Walton etc.

The perspective

- The theoretical frameworks and the scientific research methodologies intertwined with the technological means are the lenses that allow us to view the phenomenon we research, and, at the same time, constitute this phenomenon.
  - [constitute the phenomenon we research]

A note

- Consider everyday school class reality
- Temporary stable
  - Students' notes (visible) & Whiteboard (visible)
    - BUT if there is an interest to record real-time changes → invisible
    - BUT if there is a technical possibility of recording the notes as they happen → visible
- Temporary unstable
  - Dialogues (invisible)
    - BUT (ibid.)
  - Non-verbal communication (gestures, tone of voice, etc.; invisible)
    - BUT (ibid.)

The perspective

- The theoretical frameworks and the scientific research methodologies intertwined with the technological means are the lenses that allow us to view the phenomenon we research, and, at the same time, constitute this phenomenon.
  - [constitute the phenomenon we research]

MathTales!

- *What is your affective relationship with mathematics?*
  - Data collection
    - Questionnaire?
      - Closed questions // Open-ended questions
    - Interview?
      - e.g. “What is your affective relationship with mathematics?”; “Any specific incidents you may think of?” etc.
    - […]

H.F.R.I. Hellenic Foundation for Research & Innovation
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MathTales!

- *What is your affective relationship with mathematics?*
  - Digital-storytelling
    - Research project: REMEDIATE

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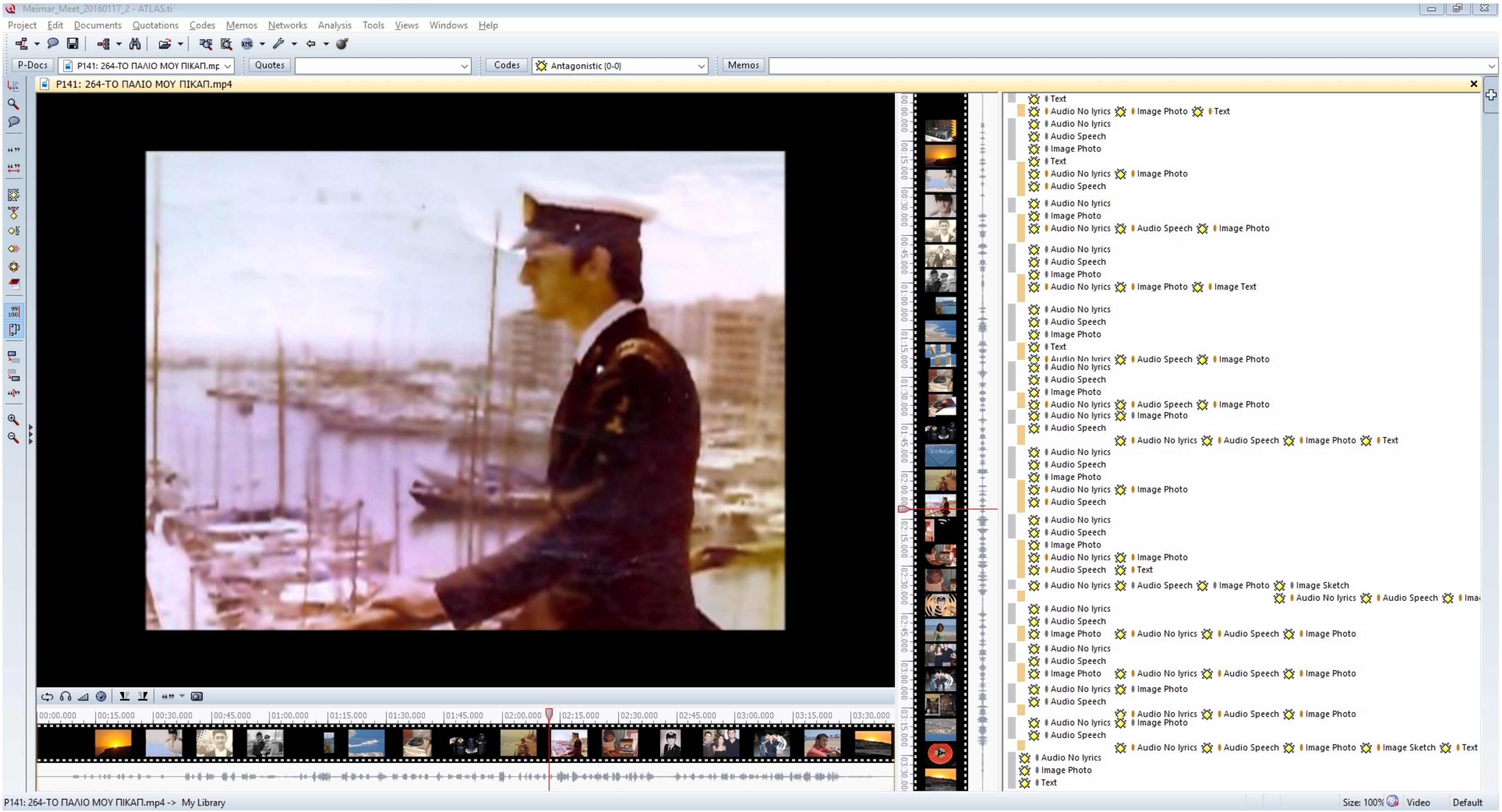

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MathTales!

- *What is your affective relationship with mathematics?*
    - Data collection
        - Questionnaire?
        - Interviews?
        - Observation?
        - Verbal? Non-verbal?
        - Digital storytelling?
            - Research project: REMEDIATE
        - Facial expressions (e.g., Moutsios-Rentzos, 2014, 2017)
        - […]

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The perspective

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A question

The capacitance C of a capacitor is the scalar physical magnitude, which is equal to the quotient of the electric charge Q of the capacitor over the electric potential V of the capacitor.

$$C = \frac{Q}{V}$$

What will happen to C, if V is doubled?

A. C will be doubled.
B. C will be half.
C. C will remain the same.
D. Other (note what): …

(Moutsios-Rentzos et al, 2019; Moutsios-Rentzos & Kalavasis, 2021)

$$y = ax, y = \frac{a}{x}, \ldots, a = \frac{x}{y}$$

$$\lambda = \frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \ldots$$

$$f(x) = ax, f(x) = \frac{a}{x}, \ldots, a = \frac{x}{f(x)}$$

$$If\ \frac{a}{b} = c, then\ \frac{a}{2b} = \frac{c}{2}$$

- The students
  - in mathematics
  - in physics

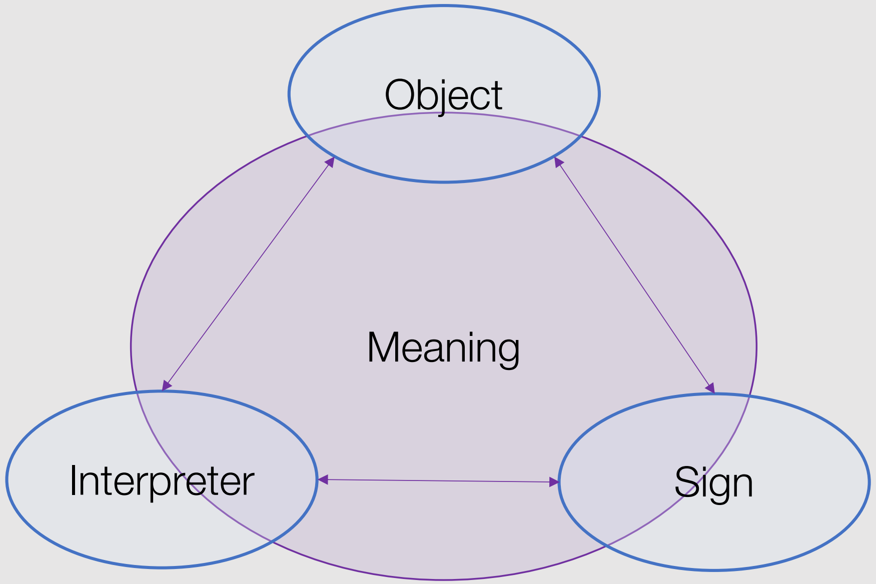
Capacitance C of a capacitor is the scalar physical magnitude, which is equal to the quotient of the electric charge Q of the capacitor over the electric potential V of the capacitor. C=Q/V (Alexakis et al., 2013, p. 32)

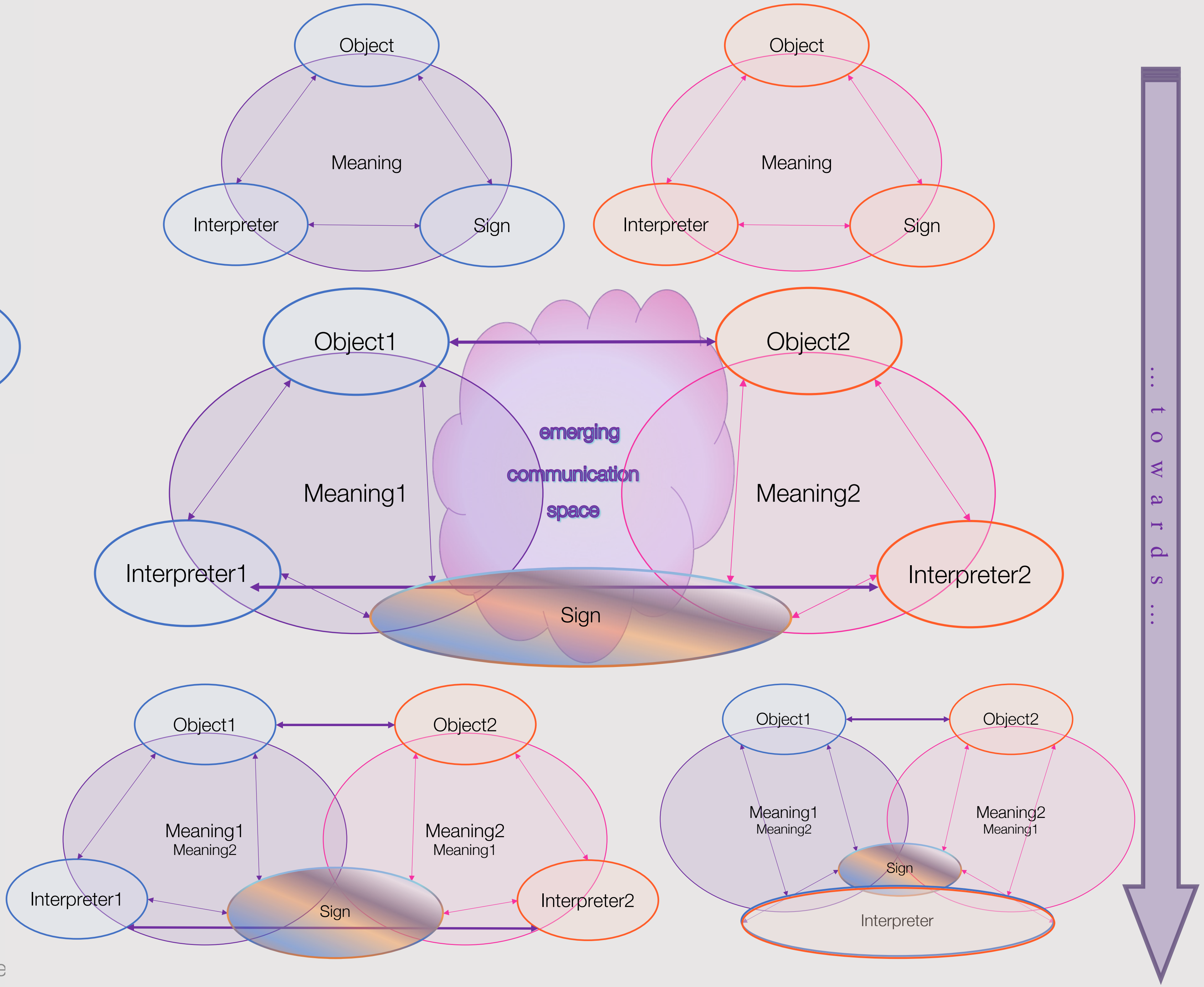
- … and a few lines lower in the same page …

The capacitance C of a capacitor does not depend on the charge or the potential, but it only depends on its shape, its dimensions and the distance between its conductors, as well as on the insulator (dielectric) between its conductors (Alexakis et al., 2013, p. 32)

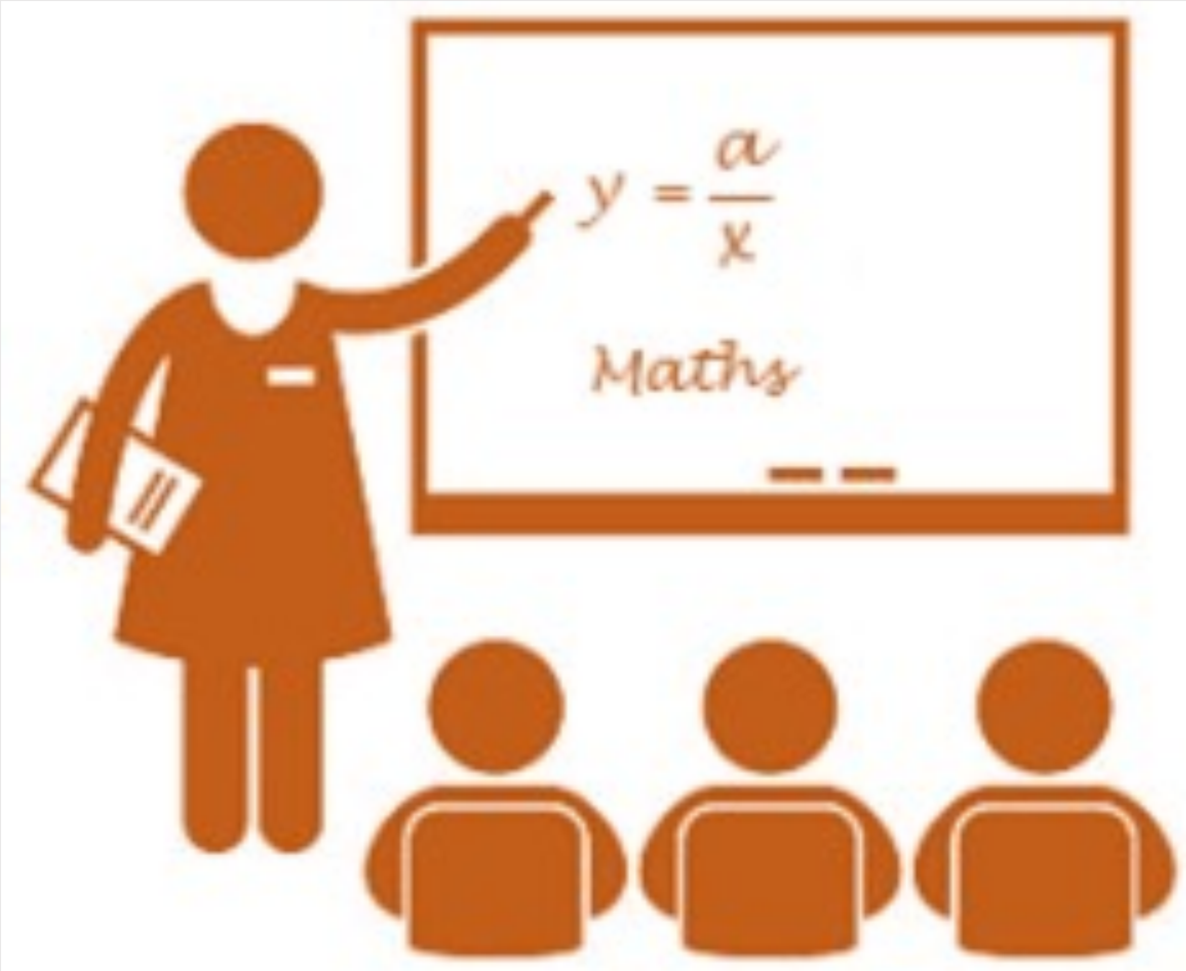
The capacitance C of a capacitor is the scalar physical magnitude, which is equal to the ~~quotient~~ *constant ratio* of the held charge over the applied electric potential. C=Q/V

- The role of familiarity with the physics notions linked (Kritikos et al, 2022)
    - velocity (U=S/t), density (p=m/V), resistance (R=V/I), capacity (C=Q/V)
- Mathematical knowledge to the rescue!?
    - … but …





(Moutsios-Rentzos, 2023; Moutsios-Rentzos et al., 2017, 2019)


y = a/x
Maths

"=" in primary school mathematics & science textbooks

- *Functions of '='*
  - *relational*
  - *operational*
  - *equivalence of measurement units*
  - *assignment of numeric value*
  - *natural language incorporation*
  - *declaration*
  - *definition*

(Moutsios-Rentzos et al, 2020)

"=" in primary school mathematics & science textbooks

- representational systems of the linked entities
  - numbers // symbols // words (natural language) // images (sketches, photos, diagrams etc) of various entities; including humans, animals, plants, everyday objects etc.
- nature of the linked entities
  - intra-disciplinary (mathematics or natural sciences)
  - extra-disciplinary
    - Mathematics // Natural sciences // Lifeworld (general)

(Moutsios-Rentzos et al, 2020)

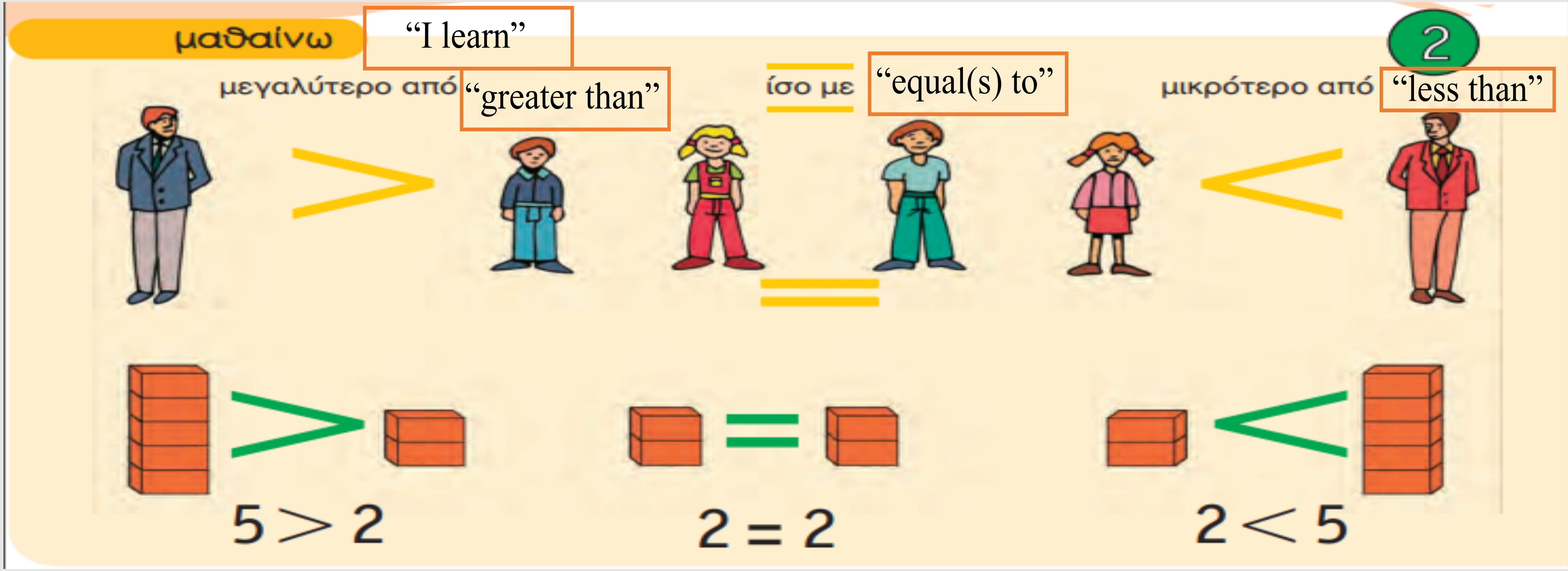
An interdisciplinary teacher-training workshop

- Interdisciplinary reflections upon mathematical (?) notation (Moutsios-Rentzos et al, 2017)

Consider the following examples on the concept of "area", from textbooks of Physics and Mathematics. Focus on the appearances of the sign "= ".

1. Can you identify uses/appearances of the sign "=" which might be a source of alternative organisations with respect to ideas of Mathematics / Physics by the students?
2. In the representations you have identified, what problems in the learning of Mathematics / Physics do you consider to come from incompatible already taught knowledge of Mathematics / Physics?
3. In the representations you identified, what problems in the learning of Mathematics / Physics do you consider to come from incompatible already taught knowledge of Physics / Mathematics?

Mathematics



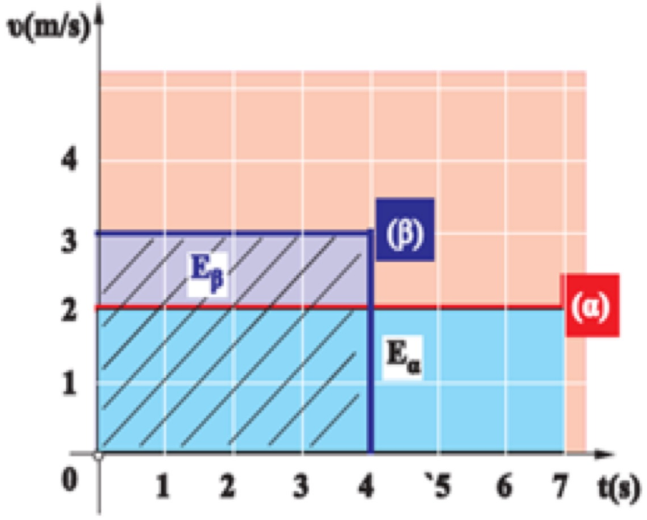


Figure 1.1.12

Velocity-time graph. The areas $E_α$ (blue) and $E_β$ (striped) give the displacements of the particles α and β, respectively.

The straight lines (α) and (β) are parallel to the time axis.

By calculating the areas $E_α$ and $E_β$ between the respective straight lines (α), (β) and the axes velocity-time, we find:

$$E_α = \text{base} \cdot \text{height} = 7s \cdot 2m/s = 14m,$$

namely the displacement of the particle α

$$\text{and } E_β = \text{base} \cdot \text{height} = 4s \cdot 3m/s = 12m,$$

namely the displacement of the particle β.

Hence, from the graph υ=f(t), we can calculate the displacement Δx, by finding the respective area that is encompassed between the axes υ, t and the straight line that represents the velocity.

Physics, Grade 10 (A Lykeio, 16 years old)

The graph of the constant acceleration in linear constantly changed motion of the car that we study, will be straight line, parallel to the time axis t, as shown in figure 1.1.19.

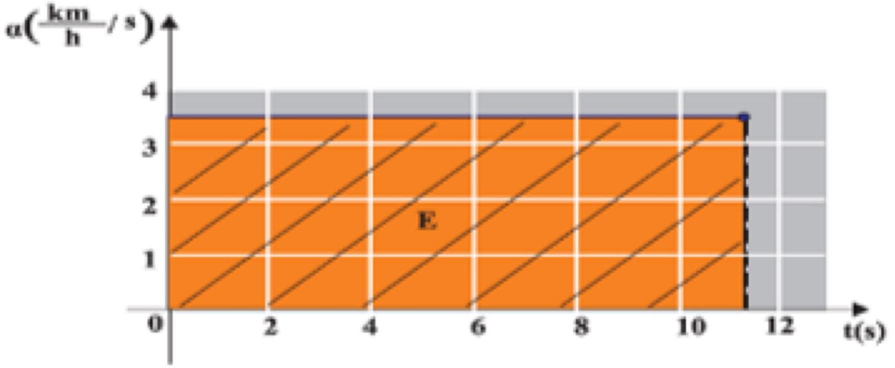


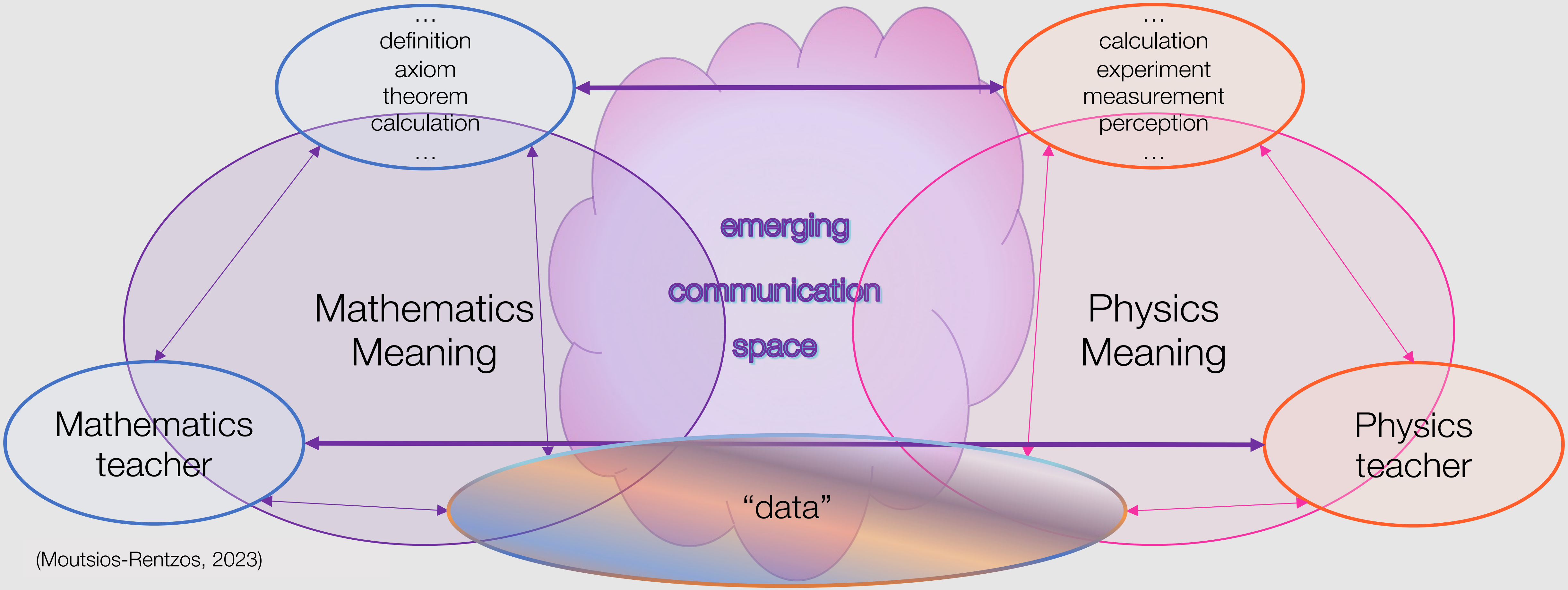
Figure 1.1.19

What could the physical meaning be of the striped area of figure 1.1.19? The area between the graph (straight line) and the axes of acceleration and time is:

$$E = \text{base} \cdot \text{height} = 3.51 \frac{km/h}{s} 11.4s = 40 km/h = υ$$

We notice that the area is arithmetically equal to the change of velocity during the 11.4s of the acceleration of the car. Therefore, the area between the straight line that represents the acceleration vs. time, is arithmetically equal to the change of velocity Δυ.

Identifying and Differentiating by Interdisciplinary Linking



(Moutsios-Rentzos, 2023)

Between the visible and the invisible

- The theoretical frameworks and the scientific methodologies intertwined with the technological means are the lenses that allow us to choose what is visible in our research, constituting the phenomena we investigate.
  - Awareness of what/who is voiced and what/who is silenced!

Concluding reflections

- Changes of lenses; perspectives and foci
  - aims (e.g. students' reasoning, teaching tool etc)
  - theoretical perspectives
  - roles
  - measures
  - technological means
    - transforming existing or changing completely (e.g. written/voice/video recordings, bio-sensors)
  - discipline(s)
    - e.g. signs and meanings (inter-/intra-); notation, reasoning, techniques, manipulations
- Systemic, interdisciplinary approaches to complexity
  - Learning as linking links
  - Interdisciplinary sources of organisations of/in mathematics
  - Identifying and Differentiating by Interdisciplinary Linking

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